Worcester County Mathematics League

Varsity Meet 2
November 29, 2017

Coaches' Copy
Rounds, Answers, and Solutions

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - November 29, 2017 ANSWER KEY

Round 1:

1. 24 (Athol)
2. $\frac{1}{6}$ or 0.167 (Quaboag)
3. $\frac{997}{495}$ (Tantasqua)

Round 2:

1. 18 (Grafton)
2. $\frac{3}{2}, 0$ or $1\frac{1}{2}, 0$ or 1.5, 0 (Tantasqua)
3. 5 (Assabet Valle

Round 3:

1. $3\sqrt{3}$ or 5.196 (Leicester)
2. 12π (Bromfield)
3. $39\frac{7}{9}$ or $\frac{358}{9}$ or ~~39.778~~ (Auburn)

   $39.\overline{7}$

Round 4:

1. $14 + 7\sqrt{2}$ (Westborough)
2. -1095 (Shepherd Hill)
3. 20 (Tahanto)

Round 5:

1. -2 (St. John's)
2. $\begin{bmatrix} 5 & 3 \\ 3 & -5 \end{bmatrix}$ (Worcester Academy)
3. 7, -15 (Hudson)

TEAM Round

1. $\frac{3}{4}$ or 0.75 (Athol)
2. 17 (Bancroft)
3. 100 (Shepherd Hill)
4. 2730 (Quaboag)
5. $\begin{bmatrix} 1 & \frac{2}{3} & 1 \\ 1 & -\frac{1}{3} & 1 \\ 4 & -\frac{8}{3} & 4 \end{bmatrix}$ (Algonquin)
6. -92,400,000 (SPM)
7. 130 (Algonquin)
8. 22.5 (Bromfield)
9. ~~$\frac{21}{28}$~~ (must be in this form) (Doherty)

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WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - November 29, 2017

Round 1: Fractions, Decimals, and Percents

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Tim had a 216 candies and ate $\frac{1}{3}$ of them. Sam then ate $\frac{1}{3}$ of the remaining candies. If Rachel eats $\frac{3}{4}$ of of what is left, how many candies remain?

2. If the length of a rectangle is one third the perimeter of the rectangle, then the width of the rectangle is what fraction of the perimeter?

3. Write $2.0\overline{14}$ as an improper, reduced fraction.

ANSWERS

(1 pt.) 1. ______________________ candies

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - November 29, 2017

Round 2: Algebra 1

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Half of a number, added to a fifth of three less than the number, is equal to two thirds of the number. Find the number.

2. The sum of two real numbers is equal to the reciprocal of the smaller number. The larger number is one greater than the smaller number. What possible values can the larger number take?

3. Jake gave Greg a 5 yard head start in a 100 yard dash and Jake ended up losing by 0.25 yards. If the runners continue at the same pace, how many more yards will Greg run until Jake catches up to him?

ANSWERS

(1 pt.) 1. ______________________

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________ yards

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - November 29, 2017

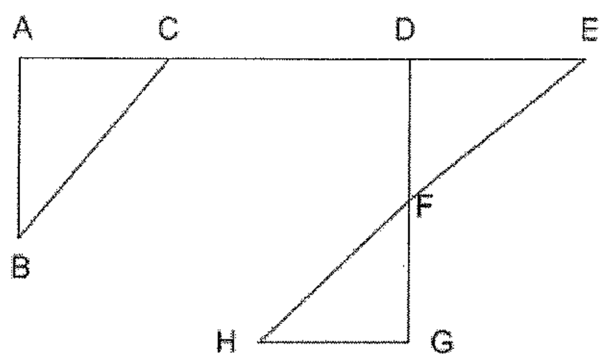
Round 3: Parallel Lines and Polygons

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Suppose that triangle ACE is equilateral with side 4 and that B and D are the midpoints of AC and CE respectively. Given that an equilateral triangle with side x has an area of $\frac{\sqrt{3}}{4}x^2$, what is the exact area of trapezoid ABDE?

2. A regular hexagon with side a has an area of $\frac{3\sqrt{3}}{2}a^2$. If a particular hexagon has an area of $54\sqrt{3}$ square inches and is inscribed in a circle, what is the circumference of the circle? Leave your answer in terms of pi.

3. 

Suppose that AB is parallel to DG, AE is parallel to HG, and BC is parallel to HE.

If AC = 24, AB = 25, BC = 36, FE = 40, and HF = 18, find DF+HG.
Note: Figure *not* drawn to scale

ANSWERS

(1 pt.) 1. ________________ inches

(2 pts.) 2. ________________ square units

(3 pts.) 3. ________________ units

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - November 29, 2017
Round 4: Sequences and Series

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Determine the exact sum of the first 6 terms of the geometric progression whose first two terms are $\sqrt{2}$ and 2.

2. 7, 4, 1, ... is the beginning of an arithmetic sequence. What is the sum of the first 30 terms of the sequence?

3. Solve for x: $\sum_{k=1}^{x}(2k+5)=520.$

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - November 29, 2017

Round 5: Matrices and Systems of Equations

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Solve for x: $\begin{vmatrix} 2x & 3 \\ x & -4 \end{vmatrix} = 22$

2. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Compute $\left(A^T \cdot A\right) - \left(A \cdot A^T\right)$.

3. Find all x such that $\begin{vmatrix} x & 9 & 6 \\ -5 & 1 & 4 \\ 3 & x & -2 \end{vmatrix} = -420$

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. $\begin{bmatrix} & \\ & \end{bmatrix}$

(3 pts.) 3. ____________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - November 29, 2017

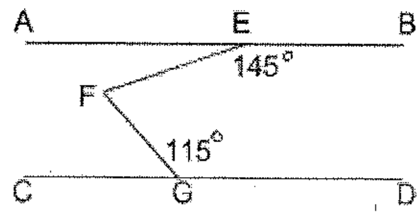
Team Round

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (2 points each)

APPROVED CALCULATORS ALLOWED

1. If $\frac{x}{3}+\frac{x}{4}+\frac{x}{6}=kx$ and $x\neq 0$ then find k.

2. Solve for x: $\sqrt{2x+2}=23-x$.



3. In the diagram to the right if AB | | CD what is the measure of angle EFG? (not drawn to scale)

4. Evaluate $4+6+10+12+16+18+22+24+...+124+126$.

5. Find a matrix A such that $\begin{bmatrix}1&2&0\\1&4&-1\\2&1&0\end{bmatrix}A=\begin{bmatrix}3&0&3\\1&2&1\\3&1&3\end{bmatrix}$

Express the elements of A as improper, reduced fractions.

6. Find the coefficient of the term a^6b^5 in the expansion of $(2a-5b)^{11}$.

7. Five of the angles of an octagon have measures which sum to $860°$. Of the remaining three angles, exactly two of them are complementary and exactly two are supplementary. Find the measure of the largest of the three remaining angles.

8. If the circumference of a circle is 100 inches, find the length of a side of a square inscribed in the circle. Round your answer to the nearest tenth of an inch.

9. The numerator and denominator of a fraction are in the ratio of 3:4. If 5 is subtracted from the numerator and 4 is subtracted from the denominator the resulting fraction is in the ration 2:3. What is the sum of the original numerator and denominator?

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 2 - November 29, 2017 - SOLUTIONS

Round 1: Fractions, Decimals, and Percents

1. Tim had a 216 candies and ate $\frac{1}{3}$ of them. Sam then ate $\frac{1}{3}$ of the remaining candies. If Rachel eats $\frac{3}{4}$ of of what is left, how many candies remain?

Solution 1: We have that

$$\left(1-\tfrac{3}{4}\right)\left(1-\tfrac{1}{3}\right)\left(1-\tfrac{1}{3}\right)216=\left(\tfrac{1}{4}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{2}{3}\times 216\right)=\left(\tfrac{1}{4}\right)\left(\tfrac{2}{3}\right)144=\tfrac{1}{6}\times 144=24.$$

Solution 2: We have that

$$216\times\tfrac{1}{4}\times\tfrac{2}{3}\times\tfrac{2}{3}=\tfrac{216}{3\times 3}=\tfrac{72}{3}=24.$$

2. If the length of a rectangle is one third the perimeter of the rectangle, then the width of the rectangle is what fraction of the perimeter?

Solution 1: If w is the rectangle's width, l is the rectangle's length and P is the rectangle's perimeter then we know that

$$P=2l+2w$$
$$P=2\left(\tfrac{1}{3}P\right)+2w$$
$$P=\tfrac{2}{3}P+2w$$
$$\tfrac{1}{3}P=2w$$
$$w=\tfrac{1}{6}P.$$

That is, the rectangle's width is one sixth of its perimeter.

Solution 2: If w is the rectangle's width, l is the rectangle's length and P is the rectangle's perimeter then we know that

$$l+w=\frac{1}{2}P$$
$$\frac{1}{3}P+w=\frac{1}{2}P$$
$$w=P\left(\frac{1}{2}-\frac{1}{3}\right)=\frac{1}{6}\text{P}.$$

3. Write $2.0\overline{14}$ as an improper, reduced fraction.

Solution 1: Let $x = 2.0\overline{14}$. Then we know that $10x = 20.\overline{14}$ and that $1000x = 2014.\overline{14}$. With this in mind, we have

$$1000x - 10x = 2014.\overline{14} - 20.\overline{14}$$
$$990x = 2014 - 20$$
$$990x = 1994$$
$$x = \frac{1994}{990} = \frac{997}{495}.$$

Solution 2: Let $x = 2.0\overline{14}$. Then we know that $100x = 201.4\overline{14}$. Therefore, we have that

$$100x = 201.4\overline{14}$$
$$-\quad x = \quad 2.0\overline{14}$$

$$99x = 199.4$$

$$\rightarrow 990x = 1994$$
$$x = \frac{1994}{990} = \frac{997}{495}.$$

Round 2: Algebra 1

1. Half of a number, added to a fifth of three less than the number, is equal to two thirds of the number. Find the number.

Solution: Let x be the number. We have that

$$\tfrac{1}{2}x + \tfrac{1}{5}(x-3) = \tfrac{2}{3}x$$
$$\tfrac{1}{2}x + \tfrac{1}{5}x - \tfrac{3}{5} = \tfrac{2}{3}x$$
$$15x + 6x - 18 = 20x$$
$$21x - 20x = 18$$
$$x = 18$$

2. The sum of two real numbers is equal to the reciprocal of the smaller number. The larger number is one greater than the smaller number. What possible values can the larger number take?

Solution 1: Let x be the larger number and y be the smaller number. We have that

$$x+y=\frac{1}{y} \qquad (1)$$
$$x-y=1 \qquad (2)$$

From equation (2) we know that $y=x-1$. Plugging this into equation (1) we have that

$$x+(x-1)=\frac{1}{x-1}$$
$$2x-1=\frac{1}{x-1}$$
$$(x-1)(2x-1)=1$$
$$2x^2-3x+1=1$$
$$2x^2-3x=0$$
$$x(2x-3)=0 \quad \rightarrow \quad x=0 \text{ or } x=1.5.$$

Solution 2: Let x be the smaller number and $x+1$ be the larger number. Then we have that

$$x+(x+1)=\frac{1}{x}$$
$$2x+1=\frac{1}{x}$$
$$2x^2+x=1$$
$$2x^2+x-1=0$$
$$x=\frac{-1\pm\sqrt{1-4(2)(-1)}}{4}=\frac{-1\pm\sqrt{9}}{4}=0.5 \text{ or } -1.$$

Since $x+1$ is the larger number, our answer is 1.5 or 0.

3. Jake gave Greg a 5 yard head start in a 100 yard dash and Jake ended up losing by 0.25 yards. If the runners continue at the same pace, how many more yards will Greg run until Jake catches up to him?

Solution 1: We can solve this problem by determining how much ground Greg loses to Jake every yard that Greg runs. Since Greg runs 95 yards and loses a total of 4.75 yards of ground to Jake, we have that Greg loses $\frac{4.75}{95}=\frac{475}{95}\times\frac{1}{100}=\frac{5}{100}=0.05$ yards of ground every yard he runs. Since Greg finishes the race with a 0.25 yard lead, Jake will catch up with Greg after Greg runs 5 more yards.

Solution 2: We have that

Name	Distance	= Rate ×	Time
Greg	95	r_1	t
Jake	99.75	r_2	t

Since the times for both runners are equal we have that

$$\frac{95}{r_1} = \frac{99.75}{r_2}$$
$$\frac{95}{99.75} = \frac{r_1}{r_2}$$

Now let x be the distance that Greg has run in total when Greg catches him. Therefore, we know that Jake will have run a total distance of $x+5$ since Greg was given a 5 yard head start. Since both runners are still running at the same rates. We have that

$$\frac{x}{x+5} = \frac{r_1}{r_2} = \frac{95}{99.75}$$
$$99.75x = 95(x+5)$$
$$99.75x = 95x + 475$$
$$4.75x = 475$$
$$x = 100.$$

If this is the total distance that Greg runs until Jake catches him and we know that he ran 95 yards in the race, Greg must run 5 more yards until Jake catches him.

Round 3: Parallel Lines and Polygons

1. Suppose that triangle ACE is equilateral with side 4 and that B and D are the midpoints of AC and CE respectively. Given that an equilateral triangle with side x has an area of $\frac{\sqrt{3}}{4}x^2$, what is the exact area of trapezoid ABDE?

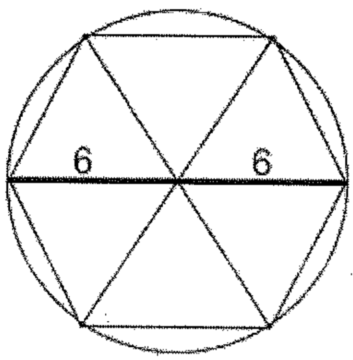
Solution: The area of an equilateral triangle with side x is equal to $\frac{\sqrt{3}}{4}x^2$. Therefore, the area of ACE is equal to $\frac{\sqrt{3}}{4}4^2 = 4\sqrt{3}$. Since B and D are midpoints, we know that AB = BC = 2. Since triangle BCD is also equilateral, we have that its area is equal to $\frac{\sqrt{3}}{4}2^2 = \sqrt{3}$.

This means that the area of trapezoid ABDE is equal to $4\sqrt{3} - \sqrt{3} = 3\sqrt{3}$.

2. A regular hexagon with side a has an area of $\frac{3\sqrt{3}}{2}a^2$. If a particular hexagon has an area of $54\sqrt{3}$ square inches and is inscribed in a circle, what is the circumference of the circle? Leave your answer in terms of pi.

Solution: Use the provided formula to determine the length of the hexagon's side:

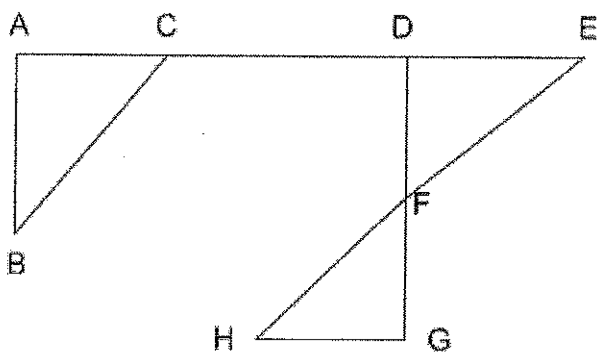
$$54\sqrt{3} = \frac{3\sqrt{3}}{2}a^2$$
$$54 = \frac{3}{2}a^2$$
$$36 = a^2 \quad \rightarrow \quad a = 6.$$



Since regular hexagons are comprised of 6 equilateral triangles, we know that each of the equilateral triangles in this regular hexagon has a side length of 6 inches. This implies that the diameter of the circle must be 12 inches as we can see in the diagram to the left.

Since a circle's circumference is equal to $d\pi$, we have that this circle's circumference is equal to 12π.

3.



Suppose that AB is parallel to DG, AE is parallel to HG, and BC is parallel to HE.

If AC = 24, AB = 25, BC = 36, FE = 40, and HF = 18, find DF+HG.
Note: Figure *not* drawn to scale

Solution: Since BC is parallel to HE, we know that triangle ABC is similar to triangle DEF. This means that

$$\frac{DF}{FE} = \frac{AB}{BC}$$
$$\frac{DF}{40} = \frac{25}{36} \quad \rightarrow \quad DF = \frac{10}{9} \times 25 = \frac{250}{9}$$

Next since AR is parallel to HG, we have that triangle FGH is similar to triangle DEF, which further implies that triangle FGH is similar to triangle ABC. Hence

$$\frac{HG}{HF} = \frac{AC}{BC}$$
$$\frac{HG}{18} = \frac{24}{36} \longrightarrow HG = 18 \times \tfrac{2}{3} = 12.$$

Therefore, we have that $DF + HG = \frac{250}{9} + 12 = \frac{250}{9} + \frac{108}{9} = \frac{358}{9}$.

Round 4: Sequences and Series

1. Determine the sum of the first 6 terms of the geometric progression whose first two terms are $\sqrt{2}$ and 2.

Solution: First determine the common ratio by dividing the second term by the first term: $r = \frac{2}{\sqrt{2}} = \sqrt{2}$.

This means the first six terms of the progression are $\sqrt{2}$, 2, $2\sqrt{2}$, 4, $4\sqrt{2}$, 8. Their sum is equal to $2 + 4 + 8 + \sqrt{2} + 2\sqrt{2} + 4\sqrt{2} = 14 + 7\sqrt{2}$.

2. 7, 4, 1, ... is the beginning of an arithmetic sequence. What is the sum of the first 30 terms of the sequence?

Solution: The common difference in this arithmetic sequence is -3. The n^{th} term in the sequence is given by $7 + (n-1)(-3)$, which means that the 30th term is $7 + 29(-3)$. Using Gauss's formula for sums of arithmetic sequences, we have that the desired sum equals

$$\frac{30}{2}[7 + 7 + 29(-3)] =$$
$$\frac{30}{2}[14 - 87] =$$
$$15 \times [-73] = -1095.$$

3. Solve for x: $\sum_{k=1}^{x}(2k+5)=520.$

Solution 1: We have that

$$\sum_{k=1}^{x}(2k+5)=520$$
$$5x+\sum_{k=1}^{x}2k=520$$
$$5x+2\sum_{k=1}^{x}k=520$$
$$5x+2\tfrac{x}{2}(x+1)=520$$
$$5x+x^2+x=520$$
$$x^2+6x-520=0$$
$$(x+26)(x-20)=0 \quad \rightarrow x=20.$$

Solution 2: Since each term in the sum is of the form $2k+5$ and k starts at 1, we have that the sum looks like,

$$7+9+11+13+\ldots=520$$

Since each term is of the form $2k+5$ and there are a total of x terms, this sum becomes

$$5x+2(1+2+3+\ldots+x)=520$$

Using Gauss's formula for the sum of the first x integers we have

$$5x+2\times\frac{x}{2}(1+x)=520$$
$$5x+x(1+x)=520$$
$$6x+x^2=520$$
$$x^2+6x-520=0$$
$$x=\frac{-6\pm\sqrt{36-4(1)(-520)}}{2}$$
$$x=\frac{-6\pm\sqrt{2116}}{2}=\frac{-6\pm 46}{2} \quad \rightarrow \quad x=\frac{40}{2}=20.$$

Round 5: Matrices and Systems of Equations

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Solve for x: $\begin{vmatrix} 2x & 3 \\ x & -4 \end{vmatrix} = 22$

Solution: This determinant simplifies to

$$2x(-4) - 3x = 22$$
$$-8x - 3x = 22$$
$$-11x = 22$$
$$x = -2$$

2. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Compute $\left(A^T \cdot A\right) - \left(A \cdot A^T\right)$.

Solution: We have that $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Therefore, $A^T \cdot A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1+9 & 2+12 \\ 2+12 & 4+16 \end{bmatrix} = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}$

Next, $A \cdot A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1+4 & 3+8 \\ 3+8 & 9+16 \end{bmatrix} = \begin{bmatrix} 5 & 11 \\ 11 & 25 \end{bmatrix}$

Hence $\left(A^T \cdot A\right) - \left(A \cdot A^T\right) = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix} - \begin{bmatrix} 5 & 11 \\ 11 & 25 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & -5 \end{bmatrix}$

3. Find all x such that $\begin{vmatrix} x & 9 & 6 \\ -5 & 1 & 4 \\ 3 & x & -2 \end{vmatrix} = -420$

Solution 1: We compute the determinant to give that

$$x(1)(-2)+(9)(4)(3)+(6)(-5)(x)-[(3)(1)(6)+x^2(4)+(-2)(-5)(9)] = -420$$
$$-2x+108-30x-[18+4x^2+90] = -420$$
$$-4x^2-32x+420 = 0$$
$$x^2+8x-105 = 0$$
$$(x-7)(x+15)=0$$

Hence $x=7$ or -15.

Solution 2: We have that

$$x\{(1)(-2)-(4)(x)\} -9\{(-5)(-2)-(4)(3)\} +6\{(-5)(x)-(1)(3)\} =-420$$
$$x\{-2-4x\} - 9\{10-12\} + 6\{-5x-3\} = -420$$
$$-2x -4x^2 -9(-2) - 30x-18) = -420$$
$$-2x - 4x^2 +18 -30x -18 = -420$$
$$-4x^2 - 32x + 420 = 0$$
$$-4\{x^2 + 8x - 105\} = 0$$
$$-4(x-7)(x+15) = 0$$

Hence $x=7$ or -15.

1. If $\frac{x}{3} + \frac{x}{4} + \frac{x}{6} = kx$ and $x \neq 0$ then find k.

Solution: Begin by multiplying each term by 12:

$$4x + 3x + 2x = 12kx$$
$$9x = 12kx$$
$$9x - 12kx = 0$$
$$x(9 - 12k) = 0$$

Since $x \neq 0$, we can divide both sides of the equation by x:

$$9 - 12k = 0$$
$$12k = 9$$
$$k = \frac{3}{4}$$

2. Solve for x: $\sqrt{2x+2} = 23 - x$.

Solution: Begin by squaring both sides of the equation:

$$\sqrt{2x+2} = 23 - x$$
$$2x + 2 = (23 - x)^2$$
$$2x + 2 = 529 - 46x + x^2$$
$$0 = 527 - 48x + x^2$$
$$0 = (x - 17)(x - 31)$$

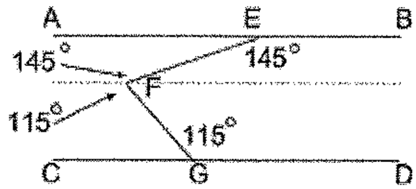
We must note however that $x \neq 31$, as it is an extraneous solution; it would imply that

$$\sqrt{64} = -8$$

Therefore, we conclude that $x = 17$.

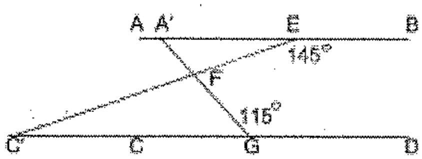
3. In the diagram to the right if AB | | CD what is the measure of angle EFG? (not drawn to scale)

Solution 1: To find the desired angle, draw in a third line which is parallel to AB and CD:



By the alternate interior angles formula and the new line drawn in the diagram to the left, we can deduce that the angle EFG is equal to $360^\circ - 145^\circ - 115^\circ = 100^\circ$.

Solution 2: We can extend segment GF to point A' and EF to point C':



Angle A'EF is 35 degrees by linear pair supplements.
Angle C'GF is 65 degrees by linear pair supplements.
Angle C'GF is an alternate interior angle to Angle FA'E

The sum of the angles of a triangle are 180 degrees.
So angle A'FE is 180 - (35 + 65) = 80 degrees.
Therefore, by linear pairs we have that angle EFG is 100 degrees.

4. Evaluate $4 + 6 + 10 + 12 + 16 + 18 + 22 + 24 + \ldots + 124 + 126.$

Solution: Notice that this sum is a sum of two separate arithmetic sequences. The first sequence is $4, 10, 16, 22, \ldots, 124$ and the second sequence is $6, 12, 18, \ldots, 126$.

Since $120 \div 6 = 20,$ we know that 124 is the 21st term of the first sequence. Likewise, since $126 \div 6 = 21,$ we know that 126 is the 21st term of the second sequence.

Hence, the sum of the terms of the first sequence is given by

$$\tfrac{21}{2}(4+124) = 1344.$$

The sum of the terms of the second sequence is given by

$$\tfrac{21}{2}(6+126) = 1386.$$

Therefore, the desired sum is equal to $1344 + 1386 = 2730.$

5. Find a matrix A such that $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 4 & -1 \\ 2 & 1 & 0 \end{bmatrix} A = \begin{bmatrix} 3 & 0 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & 3 \end{bmatrix}$

Express the terms of A as improper, reduced fractions.

Solution 1: As long as the first matrix in the equation, let's call it B, is invertible, we can solve the problem by left multiplying both sides of the equation by the inverse of the first matrix. Using the minors of the third column, we have that the determinant of the first matrix is equal to $-(-1)[1-4] = -3.$ Since the matrix has a nonzero determinant, we know that it must be invertible.

Now we need to compute the inverse of B. This can be done in four steps. First, we compute the matrix of minors:

$$\begin{bmatrix} 4\times 0 - 1(-1) & 1\times 0 - 2(-1) & 1\times 1 - 2\times 4 \\ 2\times 0 - 1\times 0 & 1\times 0 - 2\times 0 & 1\times 1 - 2\times 2 \\ 2(-1) - 4\times 0 & 1(-1) - 1\times 0 & 1\times 4 - 1\times 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -7 \\ 0 & 0 & -3 \\ -2 & -1 & 2 \end{bmatrix}$$

Next we apply the appropriate cofactors to the matrix of minors:

$\rightarrow$

Now we compute the adjugate by transposing the above matrix

$$\rightarrow \quad \begin{bmatrix} 1 & 0 & -2 \\ -2 & 0 & 1 \\ -7 & 3 & 2 \end{bmatrix}$$

Finally we multiply by the reciprocal of the determinant:

$$\rightarrow \quad B^{-1} = -\frac{1}{3}\begin{bmatrix} 1 & 0 & -2 \\ -2 & 0 & 1 \\ -7 & 3 & 2 \end{bmatrix}$$

Left multiplying both sides of the original equation by B^{-1} gives

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 4 & -1 \\ 2 & 1 & 0 \end{bmatrix} A = \begin{bmatrix} 3 & 0 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & 3 \end{bmatrix}$$

$$A = -\frac{1}{3}\begin{bmatrix} 1 & 0 & -2 \\ -2 & 0 & 1 \\ -7 & 3 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & 3 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 3-6 & -2 & 3-6 \\ -6+3 & 1 & -6+3 \\ -21+3+6 & 6+2 & -21+3+6 \end{bmatrix} =$$

$$= -\frac{1}{3}\begin{bmatrix} -3 & -2 & -3 \\ -3 & 1 & -3 \\ -12 & 8 & -12 \end{bmatrix} = \begin{bmatrix} 1 & \frac{2}{3} & 1 \\ 1 & -\frac{1}{3} & 1 \\ 4 & -\frac{8}{3} & 4 \end{bmatrix}$$

Solution 2: We will perform the same logic as in solution 1 but we will use row reduction to determine the inverse of the leftmost matrix in the equation. This is done by:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 1 & 0 \\ 0 & -3 & 0 & -2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 2 & -1 & -1 & 1 & 0 \\ 0 & 0 & -\frac{3}{2} & -\frac{7}{2} & \frac{3}{2} & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{7}{3} & -1 & -\frac{2}{3} \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 1 & 0 & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{7}{3} & -1 & -\frac{2}{3} \end{array}\right]$$

Perform the remainder of this solution as in solution 1.

6. Find the coefficient of the term a^6b^5 in the expansion of $(2a-5b)^{11}$.

Solution: Here we know that the binomial coefficient on the term a^6b^5 is going to be 11 choose 5, or simply $\binom{11}{5}$.

Therefore, the total coefficient on this term will be equal to

$$\binom{11}{5}(2^6)(-5^5) =$$

$$\tfrac{11!}{5!6!}(64)(-3125) =$$
$$(462)(64)(-3125) = \ -92,400,000.$$

7. Five of the angles of an octagon have measures which sum to 860°. Of the remaining three angles, exactly two of them are complementary and exactly two are supplementary. Find the measure of the largest of the three remaining angles.

Solution: First we know that the total measure of the angles in a polygon with n sides is equal to $(n-2)180^\circ$. In this case, that means that the measures of the 8 angles in the octagon must sum to $6 \times 180^\circ = 1080^\circ$. Since we are given that five of the angles have a sum of 860°, we know that the remaining three angles must have measures which sum to $1080^\circ - 860^\circ = 220^\circ$.

Since there are exactly two complementary and exactly two supplementary angles, we know that one angle (whose measure we will call x) must be complementary with one of the other angles and supplementary with the other.

That is, the second angle must have measure equal to $90-x$ and the third angle must have measure equal to $180-x$. Therefore, we know that

$$x+(90-x)+(180-x)=220$$
$$-x+270=220$$
$$x=50.$$

If $x=50$ then we know that the largest of the three angles has measure $180-50=130^\circ$.

8. If the circumference of a circle is 100 inches, find the length of a square inscribed in the circle. Round your answer to the nearest tenth of an inch.

Solution: If the circumference of a circle is 100 inches, we have that

$$\pi d = 100 \quad \rightarrow \quad d = \frac{100}{\pi}$$

We know that the square inscribed in the circle must have a diagonal equal to the circle's diameter, which is $\frac{100}{\pi}$.

Now let x be the side of the square. We then have that

$$x\sqrt{2} = \frac{100}{\pi}$$

$$x = \frac{100}{\pi\sqrt{2}} = \frac{100\sqrt{2}}{2\pi} = \frac{50\sqrt{2}}{\pi} = 22.5$$

9. The numerator and denominator of a fraction are in the ratio of 3:4. If 5 is subtracted from the numerator and 4 is subtracted from the denominator the resulting fraction is in the ration 2:3. What is the sum of the original numerator and denominator?

Solution: Let x be the numerator and y be the denominator. We are given that

$$\frac{x}{y} = \frac{3}{4} \qquad (1)$$

$$\frac{x-5}{y-4} = \frac{2}{3} \qquad (2)$$

From equation (1) we have that $x = \frac{3}{4}y$. Plugging this into equation (2) we have that

$$\frac{\frac{3}{4}y-5}{y-4} = \frac{2}{3}$$

$$3(\tfrac{3}{4}y - 5) = 2(y-4)$$

$$\tfrac{9}{4}y - 15 = 2y - 8$$

$$\tfrac{1}{4}y = 7 \quad \rightarrow \quad y = 28 \quad \rightarrow \quad x = 21.$$

Hence, the desired sum is $21 + 28 = 49$.